7.4 Inverse Trig Functions

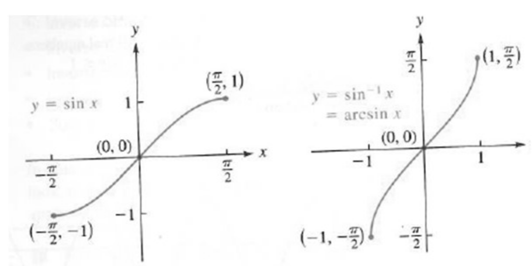
Who uses this?

Actuaries
Aerospace Engineers
Mechanical Engineers
Nuclear Engineers
Economist
Boilermakers

- For a function to have an inverse it must be one-to-one.
- One-to-one functions have to pass the horizontal line test. Each y-value can be paired with no more than one x-value. And each x-value can be paired with no more than one y-value.
- Are the graphs of the trig functions one-to-one?

Inverse Sine Function

- We can make the sine function one-to-one by restricting the domain.
- For y = sinx the domain is all real numbers and the range is [-1, 1].
- To find the inverse we are going to reflect the restricted sine graph over the line y = x.



Domain: $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$
Range: $-1 \le y \le 1$

(a) Restricted sine function

Domain: $-1 \le x \le 1$
Range: $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

(b) Inverse sine function

$y = \sin^{-1}x$

"the angle between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ with a sine of x."

Domain: $[-1,1]$

Range: $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Ex
$\sin^{-1} \frac{1}{2}$

- What angle between $\pi/2$ and $-\pi/2$ has a sine that is ½ ?

$\pi/6$

- Range of the inverse is restricted to right side of unit circle.

Find the exact values without a calculator!

$\sin^{-1}\frac{1}{2}$ $\pi/6$

$\sin^{-1} 3$ *und*

$\arcsin\left(-\frac{\sqrt{2}}{2}\right)$ $-\pi/4$

$\arcsin\left(\frac{\sqrt{3}}{2}\right)$ $\pi/3$

$\sin^{-1}(-1)$ $-\pi/2$

$\arcsin\left(-\frac{1}{2}\right)$ $-\pi/6$

Use your calculator to evaluate to 3 decimal places. (You must be in radians!!!)

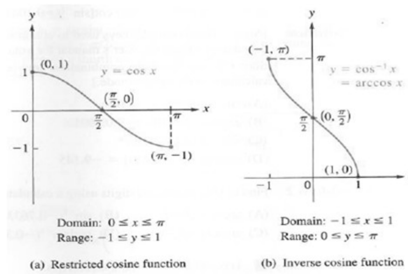
- $\sin^{-1}(0.82)$ 0.961
- $\arcsin(-0.3042)$ -0.309
- $\cot[\sin^{-1}(-0.1087)]$ -9.145

Inverse Cosine

- $y = \cos^{-1}x$ or $y = \arccos x$

Domain: $[-1,1]$

Range: $[0,\pi]$



- $y = \cos^{-1}x$ means 'the angle in the interval between 0 and pi whose cosine is x.'
- Values of the inverse cosine are located in the upper half of the unit circle.

Find the exact values without a calculator!

$\cos^{-1}\frac{1}{2}$ $\pi/3$

$\cos^{-1}(-2)$ und

$\arccos\left(-\frac{\sqrt{3}}{2}\right)$ $5\pi/6$

$\arccos\left(\frac{-\sqrt{2}}{2}\right)$ $3\pi/4$

$\cos(\cos^{-1}(0.7))$ 0.7

$\csc\left[\cos^{-1}(-0.0349)\right]$ 1.001

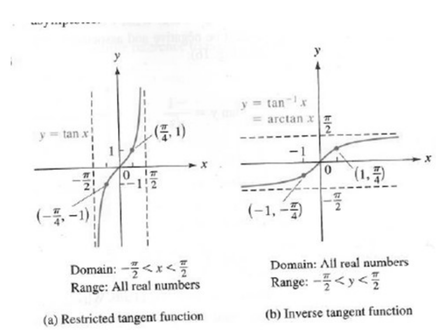
Inverse Tangent

- $y = \tan^{-1}x$ or $y = \arctan x$

Domain: $\Re$ or $(-\infty,\infty)$

Range: $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

- The vertical asymptotes become horizontal asymptotes when we reflect the graph over the line y = x.
- The values are on the right side of the unit circle.



Find the exact values without a calculator!

$\tan^{-1} 1$ $\quad \pi/4$

$\tan^{-1}\sqrt{3}$ $\quad \pi/3$

$\arctan\left(-\frac{\sqrt{3}}{3}\right)$ $\quad -\pi/6$

$\arctan(-20)$ $\quad -1.521$

$\arctan(\tan\frac{\pi}{2})$ $\quad und$

$\sec\left[\tan^{-1}(-0.1308)\right]$ $\quad 1.009$

Domain of Compositions of Trig Functions

$$f[f^{-1}(x)] = x$$

If $-1 \le x \le 1$, then $\sin(\sin^{-1} x) = x$

If $-1 \le x \le 1$, then $\cos(\cos^{-1} x) = x$

If $-\infty \le x \le \infty$, then $\tan(\tan^{-1} x) = x$

$$f^{-1}[f(x)] = x$$

If $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$, then $\sin^{-1}(\sin x) = x$

If $0 \le x \le \pi$, then $\cos^{-1}(\cos x) = x$

If $-\frac{\pi}{2} < x < \frac{\pi}{2}$, then $\tan^{-1}(\tan x) = x$

Find the exact value, if it exists.

$\sin\left(\sin^{-1}\left(\frac{1}{2}\right)\right)$ $\quad \frac{1}{2}$

$\sin^{-1}\left(\sin\frac{5\pi}{4}\right)$ $\quad -\frac{\pi}{4}$

$\arctan\left(\tan\frac{\pi}{2}\right)$ $\quad und$

$\arcsin\left(\sin\frac{2\pi}{3}\right)$ $\quad \frac{\pi}{3}$

Find the exact value if it exists.

$\sin\left(2\cos^{-1}\frac{\sqrt{2}}{2}\right)$

$\sin(2(45°))$

$\sin(90°)$

1

$\cos\left(\cos^{-1}0 + \sin^{-1}\frac{1}{2}\right)$

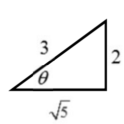
$\cos(90° + 30°)$

$\cos(120°)$

$-\frac{1}{2}$

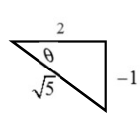
Find the exact value without a calculator.

$\cos\left(\sin^{-1}\frac{2}{3}\right)$



$\cos\theta = \frac{\sqrt{5}}{3}$

$\tan\left[\sin^{-1}\left(-\frac{1}{\sqrt{5}}\right)\right]$



$\tan\theta = -\frac{1}{2}$

Write as an algebraic expression in x.

- $\sin(\cos^{-1}x)$

$u = \cos^{-1} x$

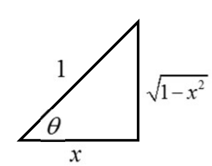
$$\sin u = \pm\sqrt{1-\cos^2 u} = \sqrt{1-\cos^2(\cos^{-1} x)}$$

$$= \sqrt{1-\left(\cos(\cos^{-1} x)\right)^2} = \sqrt{1-x^2}$$

- OR

$Let\, \theta = \cos^{-1} x$

$\cos\theta = x$



Write as an algebraic expression in x.

- $\tan(\cos^{-1}x)$

$u = \cos^{-1} x$

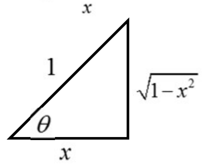
$$\tan u = \frac{\sin u}{\cos u} = \frac{\sqrt{1-\cos^2 u}}{\cos u} = \frac{\sqrt{1-\cos^2\left(\cos^{-1}x\right)}}{\cos\left(\cos^{-1}x\right)}$$
$$= \frac{\sqrt{1-x^2}}{x}$$

- OR

$Let\, \theta = \cos^{-1} x$

$\cos\theta = x$

$\tan\theta = \frac{\sqrt{1-x^2}}{x}$



Write as an algebraic expression in x.

- $\cos(\arcsin x)$

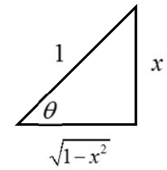
$u = \sin^{-1} x$

$$\cos u = \pm\sqrt{1-\sin^2 u} = \sqrt{1-\sin^2\left(\sin^{-1}x\right)}$$
$$= \sqrt{1-x^2}$$

- OR

$Let\, \theta = \sin^{-1} x$

$\sin\theta = x$



7.4 b Inverse Trig Functions

Sec, csc, cot

- $y = \sin^{-1}x$

  Domain: $[-1,1]$

  Range: $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

- $y = \cos^{-1}x$

  Domain: $[-1,1]$

  Range: $[0,\pi]$

- $y = \csc^{-1}x$

  Domain: $(-\infty,-1]\cup[1,\infty)$

  Range: $\left[-\frac{\pi}{2},\frac{\pi}{2}\right], \boxed{y \neq 0}$

- $y = \sec^{-1}x$

  Domain: $(-\infty,-1]\cup[1,\infty)$

  Range: $[0,\pi], \boxed{y \neq \frac{\pi}{2}}$

- $y = \tan^{-1}x$

  Domain: $(-\infty,\infty)$

  Range: $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

- $y = \cot^{-1}x$

  Domain: $(-\infty,\infty)$

  Range: $(0,\pi)$

Find the exact values without a calculator!

$\csc^{-1}\frac{1}{2}$ $\quad und$

$\csc^{-1}(-2)$ $\quad -\pi/6$

$\operatorname{arcsec}\left(-\sqrt{2}\right)$ $\quad 3\pi/4$

$\operatorname{arcsec}\left(\frac{2\sqrt{3}}{3}\right)$ $\quad \pi/6$

$\cot^{-1}(-1)$ $\quad 3\pi/4$

$\operatorname{arccot}(0)$ $\quad \pi/2$

Use your calculator to evaluate to 3 decimal places. (You must be in radians!!!)

- $\csc^{-1}(1.82)$ $\quad 0.582$
- $\operatorname{arcsec}(-5.3042)$ $\quad 1.760$
- $\tan[\cot^{-1}(-3.1087)]$ $\quad -0.322$

Find the exact value, if it exists.

$\sin\left(\sec^{-1}(2)\right)$ $\quad \dfrac{\sqrt{3}}{2}$

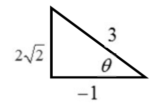
$\sin^{-1}\left(\csc\dfrac{5\pi}{4}\right)$ $\quad und$

$\arctan\left(\cot\dfrac{\pi}{2}\right)$ $\quad 0$

$\operatorname{arccot}\left(\sin\dfrac{3\pi}{2}\right)$ $\quad \dfrac{3\pi}{4}$

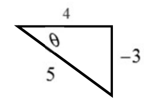
Find the exact value without a calculator.

$\tan\left(\sec^{-1}(-3)\right)$



$\tan\theta = -2\sqrt{2}$

$\cot\left(\csc^{-1}\left(-\dfrac{5}{3}\right)\right)$



$\cot\theta = -\dfrac{4}{3}$

7.5 Trig Equations

An equation that contains trig functions is called a trig equation.

To solve a trig equation we find ALL VALUES of the variable that make the equation true.

Ex

2sinx – 1 = 0

$$+1 \qquad +1$$

$$\frac{2\sin x}{2} = \frac{1}{2}$$

$$\sin x = \frac{1}{2}$$

$$x = \sin^{-1}\frac{1}{2} = \frac{\pi}{6}, \frac{5\pi}{6}$$

$$x = \frac{\pi}{6} + 2\pi k$$

$$x = \frac{5\pi}{6} + 2\pi k$$

$$k \in \mathbb{Z}$$

Ex

$\tan^2$x - 3 = 0

$$+3 \qquad +3$$

$$\tan^2 x = 3$$

$$\tan x = \pm\sqrt{3}$$

$$x = \tan^{-1}\pm\sqrt{3} = \frac{\pi}{3}, \frac{2\pi}{3}$$

$$x = \frac{\pi}{3} + \pi k$$

$$x = \frac{2\pi}{3} + \pi k$$

$$k \in \mathbb{Z}$$

Ex

2sin^{2}x – 13sinx + 6 = 0

$$(2\sin x - 1)(\sin x - 6) = 0$$

$$2\sin x - 1 = 0$$

$$+1 \quad +1$$

$$\frac{2\sin x}{2} = \frac{1}{2}$$

$$\sin x = \frac{1}{2}$$

$$x = \frac{\pi}{6}, \frac{5\pi}{6}$$

$$\sin x - 6 = 0$$

$$+6 \quad +6$$

$$\sin x = 6$$

$$und$$

$$x = \frac{\pi}{6} + 2\pi k$$

$$x = \frac{5\pi}{6} + 2\pi k$$

$$k \in \mathbb{Z}$$

Ex

$2\cos^2 x - 7\cos x + 3 = 0$

$$(2\cos x - 1)(\cos x - 3) = 0$$

$$2\cos x - 1 = 0 \qquad \cos x - 3 = 0$$

$$+1 \quad +1 \qquad\qquad +3 \quad +3$$

$$\frac{2\cos x}{2} = \frac{1}{2} \qquad \cos x = 3$$

$$und$$

$$\cos x = \frac{1}{2}$$

$$x = \frac{\pi}{3}, \frac{5\pi}{3}$$

$$x = \frac{\pi}{3} + 2\pi k$$

$$x = \frac{5\pi}{3} + 2\pi k \qquad k \in \mathbb{Z}$$

Ex

$$\frac{\sin x}{\cos x} = \frac{\cos x}{\cos x}$$

$$\frac{\sin x}{\cos x} = 1$$

$$\tan x = 1$$

$$x = \tan^{-1} 1 = \frac{\pi}{4}, \frac{5\pi}{4}$$

$$x = \frac{\pi}{4} + \pi k \qquad k \in \mathbb{Z}$$

Ex

$3\cos x = 2\sin^2 x$

$$3\cos x = 2(1 - \cos^2 x)$$

$$2\cos^2 x + 3\cos x - 2 = 0$$

$$(2\cos x - 1)(\cos x + 2) = 0$$

$$2\cos x - 1 = 0 \qquad \cos x + 2 = 0$$

$$\cos x = \frac{1}{2} \qquad \cos x = -2$$

$$und$$

$$x = \frac{\pi}{3}, \frac{5\pi}{3}$$

$$x = \frac{\pi}{3} + 2\pi k$$

$$x = \frac{5\pi}{3} + 2\pi k \qquad k \in \mathbb{Z}$$

Ex

$1 + \sin x = 2\cos^2 x$

$$1 + \sin x = 2(1 - \sin^2 x)$$

$$2\sin^2 x + \sin x - 1 = 0$$

$$(2\sin x - 1)(\sin x + 1) = 0$$

$$2\sin x - 1 = 0 \qquad \sin x + 1 = 0$$

$$\sin x = \frac{1}{2} \qquad \sin x = -1$$

$$x = \frac{\pi}{6}, \frac{5\pi}{6} \qquad x = -\frac{\pi}{2}$$

$$x = \frac{\pi}{6} + 2\pi k \qquad x = -\frac{\pi}{2} + 2\pi k$$

$$x = \frac{5\pi}{6} + 2\pi k \qquad k \in \mathbb{Z}$$

Ex 1

$\sin 2x - \cos x = 0$

$$2\sin x \cos x - \cos x = 0$$

$$\cos x(2\sin x - 1) = 0$$

$$2\sin x - 1 = 0 \qquad \cos x = 0$$

$$\sin x = \frac{1}{2} \qquad x = \frac{\pi}{2}, \frac{3\pi}{2}$$

$$x = \frac{\pi}{6}, \frac{5\pi}{6}$$

$$x = \frac{\pi}{6} + 2\pi k \qquad x = \frac{\pi}{2} + \cancel{2}\pi k$$

$$x = \frac{5\pi}{6} + 2\pi k \qquad \cancel{x = \frac{3\pi}{2} + 2\pi k} \qquad k \in \mathbb{Z}$$

Ex 2

$3\tan^3 x - \tan x = 0$

$$\tan x(3\tan^2 x - 1) = 0$$

$$\tan x = 0 \qquad 3\tan^2 x - 1 = 0$$

$$x = 0, \pi \qquad \tan x = \pm\frac{1}{\sqrt{3}}$$

$$x = \frac{\pi}{6}, \frac{5\pi}{6}$$

$$x = 0 + \pi k \qquad x = \frac{\pi}{6} + \pi k$$

$$x = \frac{5\pi}{6} + \pi k \qquad k \in \mathbb{Z}$$

Ex 3
secxtanx = 4sinx

$$\frac{1}{\cos x}\cdot\frac{\sin x}{\cos x}=4\sin x$$
$$\frac{\sin x}{1-\sin^2 x}=4\sin x$$
$$\sin x=4\sin x-4\sin^3 x$$
$$4\sin^3 x-3\sin x=0$$
$$\sin x(4\sin^2 x-3)=0$$

$$\sin x=0 \qquad 4\sin^2 x-3=0$$
$$x=0,\pi \qquad \sin^2 x=\frac{3}{4}$$
$$\sin x=\pm\frac{\sqrt{3}}{2}$$
$$x=\frac{\pi}{3},\frac{2\pi}{3},\frac{4\pi}{3},\frac{5\pi}{3}$$
$$x=0+\pi k \qquad x=\frac{\pi}{3}+\pi k$$
$$x=\frac{2\pi}{3}+\pi k$$

$$x=0+\frac{\pi}{3}k \qquad k\in\mathbb{Z}$$

Ex 4
cosx + 1 = sinx

$$\cos^2 x+2\cos x+1=\sin^2 x$$
$$\cos^2 x+2\cos x+1=1-\cos^2 x$$
$$2\cos^2 x+2\cos x=0$$
$$(2\cos x)(\cos x+1)=0$$

$$2\cos x=0 \qquad \cos x+1=0$$
$$x=\frac{\pi}{2},\cancel{\frac{3\pi}{2}} \qquad \cos x=-1$$
$$x=\pi$$

$$x=\frac{\pi}{2}+\pi k \qquad x=\pi+2\pi k$$
$$k\in\mathbb{Z}$$

Ex 5
2sin3x – 1 = 0 [0,2π]

$$\sin 3x=\frac{1}{2}$$
$$3x=\frac{\pi}{6},\frac{5\pi}{6}$$
$$3x=\frac{\pi}{6}+2\pi k \qquad 3x=\frac{5\pi}{6}+2\pi k$$
$$x=\frac{\pi}{18}+\frac{2\pi k}{3} \qquad x=\frac{5\pi}{18}+\frac{2\pi k}{3}$$
$$x=\frac{\pi}{18},\frac{13\pi}{18},\frac{25\pi}{18} \qquad x=\frac{5\pi}{18},\frac{17\pi}{18},\frac{29\pi}{18}$$

Ex 5
$\sqrt{\mathbf{3}}\tan\frac{\boldsymbol{x}}{\mathbf{2}}-1=0$

$$\tan\frac{x}{2}=\frac{\sqrt{3}}{3}$$
$$\frac{x}{2}=\frac{\pi}{6},\frac{7\pi}{6}$$
$$\frac{x}{2}=\frac{\pi}{6}+\pi k$$
$$x=\frac{\pi}{3}+2\pi k \qquad k\in\mathbb{Z}$$